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ON THE OPTIMAL STABILIZATION OF CONTROLLED SYSTEMS

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Stabilization of the steady motion of a system by additional forces with minimization of a certain functional characterizing control quality [1] is considered. The problem of determining the form of the integrand in the quality criterion and of the controlling forces from a certain class in such a way that the Liapunov function for the uncontrolled system can serve as the Liapunov function for the same system under the action of additional controlling forces is investigated. This problem is close to the inversion problem of analytical regulator construction [2]. The problem of optimal stabilization in some of the parameters [3] is stated and a theorem generalizing the basic theorem on stabilization in all the variables [1] is proved. Both problems are considered with specific reference to mechanical systems with a generalized energy integral of fixed sign. The results are illustrated by means of several examples. These include the problem of optimal stabilization of the positions relative to equilibrium and of the steady motions of a gyrostat satellite.

1. Let us consider the equations of perturbed motion of some system

$$\frac{dx_s}{dt} = X_s(t, x_1, \ldots, x_n) \qquad (s = 1, \ldots, n) \tag{1.1}$$

whose right sides X_s are defined in the domain

$$t \geqslant t_0, \quad |x_s| \leqslant H, \quad H = \text{const} > 0 \quad (s = 1, \ldots, n) \tag{1.2}$$

We assume that the functions X_s in domain (1.2) are continuous and that they satisfy the conditions which ensure the existence and uniqueness of the solutions of Eqs. (1.1) under any initial conditions from the domain (1.2); we also assume fulfillment of the identities

$$X_s(t, 0, \ldots, 0) \equiv 0 \quad (s = 1, \ldots, n)$$

Let us suppose that Eqs. (1.1) are associated with a continuous and unique function $V(t, x_1, \ldots, x_n)$ which vanishes for $x_s = 0$. This function is positive-definite and can have an infinitely small upper bound. By virtue of Eqs. (1.1) its derivative with respect to time, i.e.

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} X_s = W(t, x_1, \ldots, x_n)$$

is either negative-definite, or negative-constant, or identically equal to zero.

Under these conditions the unperturbed motion $x = 0$ is uniformly stable in t_0. Moreover, in the case of a negative-definite function $W(t, x)$ the unperturbed motion is asymptotically stable as in the case of a negative-constant function $W(x)$ if the right sides of Eqs. (1.1) do not depend on time and if the manifold $W(x) = 0$ does not contain entire motions of system other than the motion $x = 0$ [4].

There have been frequent attempts to make the unperturbed motion $x = 0$ asymptotically stable in such a way as to minimize the integral

$$J = \int_{t_0}^{\infty} \omega(t, x_1[t], \ldots, x_n[t]; u_1[t], \ldots, u_r[t])\, dt \tag{1.3}$$

characterizing the quality of the transient process for all initial conditions from the domain

$$t \geqslant t_0, \quad |x_s| \leqslant H_1 < H \quad (s = 1, \ldots, n) \tag{1.4}$$

by subjecting the system to a system of additional forces of the form $Y_s(t, x_1, \ldots, x_n; u_1, \ldots, u_r)$ defined in domain (1.2).

The integrand $\omega(t, x, u)$ in (1.3) is a certain continuous nonnegative function defined in domain (1.2). The controlling forces $u_j = u_j(t, x_1, \ldots, x_n)$ must be defined and continuous in domain (1.2) and satisfy the equations

$$u_j(t, 0, \ldots, 0) = 0 \quad (j = 1, \ldots r)$$

We also assume that the right sides of the system of equations

$$\frac{dx_s}{dt} = X_s(t, x_1, \ldots, x_n) + Y_s(t, x_1, \ldots x_n; u_1, \ldots, u_r) \tag{1.5}$$

satisfy the conditions of existence and uniqueness of the solutions in domain (1.2); moreover, $Y_s(t, 0, \ldots 0; 0, \ldots, 0) \equiv 0$.

The symbols $u_j[t]$ represent the magnitudes of the controlling forces $u_j[t] = u_j(t, x_1[t], \ldots, x_n[t])$ as functions of the time alone which are realized in system (1.5) for $u_j = u_j(t, x_1, \ldots, x_n)$; the symbols $x_s[t]$ represent those motions of system (1.5) which are generated by the control $u_j[t]$.

The solution of this problem, i.e. the controlling forces $u_j = u_j^0(t, x)$ which ensure the asymptotic stability of the motion $x = 0$ by virtue of Eqs. (1.5) and minimize integral (1.3), can be determined with the aid of the familiar basic theorem [1] on optimal stabilization which is a modification of the Liapunov theorem on asymptotic stability with allowance for certain considerations of Bellman's dynamic programing method. As we know, the problem is reducible to the determination of the Liapunov optimal function $V^\circ(t, x)$ and of the optimal controlling forces $u_j^\circ(t, x)$; the former satisfies a partial differential equation which must be solved with allowance for a single additional inequality. This constitutes a quite difficult problem [1].

In this connection we pose the question of the form which the known function

$\omega(t;\ x;\ u)$ in integral (1.3) for the uncontrolled system under consideration must take in order for the Liapunov function $V(t,\ x)$ which solves the problem of the stability of the trivial solution of system (1.1) to serve as the optimal Liapunov function $V^\circ(t,\ x)$ for the same system under the action of the additional controlling forces $Y_s(t;\ x;\ u)$ when the equations of perturbed motion are of the form (1.5).

It is clear that this question is closely related to the inversion of the problem of analytical construction of regulators, and also to the problem of selecting an optimizing functional [2].

For simplicity we shall limit our investigation to the class of functions $\omega(t;\ x;\ u)$ of the following structure:

$$\omega(t; x; u) = F(t, x_1, \ldots, x_n) + S,\ S = \sum_{i,j=1}^{r} \beta_{ij} u_i u_j \tag{1.6}$$

where $F(t,\ x)$ is a nonnegative function to be determined and S is a given positive-definite quadratic form with symmetric coefficients $\beta_{ij} = \beta_{ji}$, and to the class of additional forces $Y_s(t;\ x;\ u)$ linear with respect to the controlling forces,

$$Y_s(t; x; u) = \sum_{j=1}^{r} m_{sj}(t, x_1, \ldots, x_n) u_j \tag{1.7}$$

Let us find the time derivative of the Liapunov function $V(t,\ x)$ known for system (1.1) by virtue of system (1.5) with allowance for (1.7)

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{s=1}^{n} \frac{\partial V}{\partial x_s}(X_s + Y_s) = W(t, x) + \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} \sum_{j=1}^{r} m_{sj} u_j \tag{1.8}$$

and construct the following expression [1] with allowance for (1.6):

$$B[V; t, x; u] = W(t, x) + \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} \sum_{j=1}^{r} m_{sj} u_j + F(t, x) + \sum_{ij=1} \beta_{ij} u_i u_j \tag{1.9}$$

By the conditions of the optimal stabilization theorem this expression reaches its minimum value of zero for $u_j = u_j^\circ$. The optimal controlling forces satisfy the equations

$$\frac{\partial B}{\partial u_j} = \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} m_{sj} + 2 \sum_{i=1}^{r} \beta_{ij} u_i^\circ = 0 \quad (j = 1, \ldots, r) \tag{1.10}$$

Solving Eqs. (1.10), we obtain

$$u_j^\circ(t, x) = -\frac{1}{2} \sum_{k=1}^{r} \frac{\Delta_{kj}}{\Delta} \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} m_{ik} \quad (j = 1, \ldots, r) \tag{1.11}$$

Here Δ_{kj} is the algebraic complement of the element β_{kj} of the determinant $\Delta = \| \beta_{ij} \| > 0$.

Since the terms dependent on u_j in expression (1.9) can be expressed as

$$\sum_{s=1}^{n} \frac{\partial V}{\partial x_s} \sum_{j=1}^{r} m_{sj} u_j + \sum_{i,j=1}^{r} \beta_{ij} u_i u_j = \sum_{i,j=1}^{r} \beta_{ij}(u_i - u_i^\circ)(u_j - u_j^\circ) - \sum_{i,j=1}^{r} \beta_{ij} u_i^\circ u_j^\circ$$

with allowance for (1.11), it clearly follows that the expression for $B[V;\ t,\ x;\ u]$ with respect to u_j attains its minimum value for $u_j = u_j^\circ$.

Substituting values (1.11) in place of u_j in expression (1.9) and equating the result to zero, we obtain the equation

$$B[V;\ t,\ x;\ u^\circ] = 0$$

from which we obtain the function

$$F(t,x) = -W(t,x) + \sum_{i,j=1}^{r} \beta_{ij} u_i^\circ u_j^\circ \tag{1.12}$$

If the function $W(t, x)$ is negative definite, the function $F(t, x)$ is positive-definite; on the other hand, if the function $W(t, x) \leqslant 0$, the function $F(t, x)$ is generally positive-constant.

From Eq. (1.12) we see that the function $F(t, x)$ depends not only on the given Liapunov function $V(t, x)$ and its derivative $W(t, x)$, but also on the coefficients of the form S (1.6) and on the elements of the matrix $\| m_{sj} \|$ of additional forces (1.7). In other words, it cannot be determined uniquely from the functions V and W. Only in the particular case where the coefficients m_{sj} are such that

$$\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} m_{ik} = 0 \qquad (k = 1, \ldots, r) \tag{1.13}$$

do we have

$$F(t, x) = -W(t, x) \tag{1.14}$$

In this case formulas (1.11) imply that the controlling forces

$$u_j^\circ(t, x) = 0 \qquad (j = 1, \ldots, r)$$

We have thus established the structure of the function $F(t, x)$ occurring in (1.6), so that quality criterion (1.3) becomes

$$J = \int_{t_0}^{\infty} \Big(-W(t, x[t]) + \sum_{i,j=1}^{r} \beta_{ij} u_i^\circ u_j^\circ + \sum_{i,j=1}^{r} \beta_{ij} u_i u_j \Big) dt \tag{1.15}$$

Positive-definite function (1.12) in quality criterion (1.3), (1.15) ensures a specific law of decay of the motions $x_s[t]$; the solution of the optimal stabilization problem is therefore sufficiently simple and obtainable in closed form. These facts in a certain sense justify [1] the choice of class (1.6), (1.7).

By virtue of system (1.5) with allowance for (1.7), (1.11), (1.15), the time derivative of $V(t, x)$ is given by

$$\frac{dV}{dt} = W(t,x) - 2 \sum_{i,j=1}^{r} \beta_{ij} u_i^\circ u_j^\circ \tag{1.16}$$

All of our statements concerning the sign of the function $F(t, x)$ are also valid for the sign of the function $-dV/dt$.

Hence, if the function $W(t, x)$ is negative-definite, then controlling forces (1.11) ensure the asymptotic stability of the unperturbed motion $x = 0$ by virtue of Eqs. (1.5) (as in the case where the function $W(x)$ is negative-constant or identically equal to zero); moreover, the right sides of Eqs. (1.5) do not depend explicitly on time, provided the manifold M of points where

$$W(x) - 2 \sum_{i,j=1}^{r} \beta_{ij} u_i^\circ u_j^\circ = 0 \tag{1.17}$$

does not contain entire motions of system (1.5) provided that $x = 0$ [4]. In both cases we have

$$\int_{t_0}^{\infty} \Big(F(t, x^0[t]) + \sum_{i,j=1}^{r} \beta_{ij} u_i^\circ [t]\, u_j^\circ [t] \Big) dt = \min \int_{t_0}^{\infty} \Big(F(t, x[t]) +$$

$$+\sum_{i,j=1}^{r} \beta_{ij} u_i [t] u_j [t] \Big) dt = V(t_0, x(t_0)) \tag{1.18}$$

where the function $F(t, x)$ is defined by Eq. (1.12).

We note that the results remain valid as in the case of a scalar control when

$$u_j = u \quad (j=1,\ldots,r), \qquad \sum_{j=1}^{r} m_{ij} = m_i, \qquad \sum \beta_{ij} = \beta > 0$$

Here we have just one equation of the form (1.10), which instead of (1.11) yields

$$u^\circ = -\frac{1}{2\beta} \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} m_s \tag{1.19}$$

so that expression (1.12) becomes

$$F(t, x) = -W(t, x) + \frac{1}{4\beta} \Big(\sum_{s=1}^{n} \frac{\partial V}{\partial x_s} m_s \Big)^2$$

Quality criterion (1.15) can be written as

$$J = \int_{t_0}^{\infty} \Big[-W(t, x) + \frac{1}{4\beta} \Big(\sum_{s=1}^{n} \frac{\partial V}{\partial x_s} m_s \Big)^2 + \beta u^2 \Big] dt \tag{1.20}$$

and manifold (1.17) as

$$W(x) - \frac{1}{2\beta} \Big(\sum_{s=1}^{n} \frac{\partial V}{\partial x_s} m_s \Big)^2 = 0 \tag{1.21}$$

We have thus proved the following theorem.

Theorem 1.1. If a positive-definite function V with an infinitely small upper bound is known for stable system (1.1), then this function is the optimal Liapunov function for system (1.5), (1.7) optimized by controlling forces (1.11) or (1.19) with respect to functional (1.15) or (1.20) in cases when the function W is negative-definite or when $W \leqslant 0$. The right sides of system (1.5) do not explicitly depend on time, and manifold (1.17) or (1.21) does not contain entire motions of system (1.15) other than $x = 0$.

Corollary. If there exists at least one nonzero matrix which satisfies condition (1.13), then system (1.1) whose asymptotic stability has been established by way of some Liapunov function V can be regarded as the optimal system, i.e. as the solution of the problem of analytical construction for system (1.5), (1.7) optimized with respect to the functional [2]

$$J = \int_{t_0}^{\infty} \Big[-W(t, x) + \sum_{i,j=1}^{r} \beta_{ij} u_i u_j \Big] dt$$

In the case where the manifold M defined by Eqs. (1.17) or (1.21) contains entire motions of system (1.5) in addition to $x = 0$, controlling forces (1.11) ensure simple stability of the motion $x = 0$ by virtue of Eqs. (1.5); moreover,

$$V(t_0, x(t_0)) = \int_{t_0}^{\infty} \Big(F(t, x^\circ [t]) + \sum_{i,j=1}^{r} \beta_{ij} u_i^\circ [t] u_j^\circ [t] \Big) dt +$$

$$+ V(t, x^\circ [t])_{t=\infty} = \min \Big(\int_{t_0}^{\infty} \Big(F(t, x[t]) + \sum_{i,j=1}^{r} \beta_{ij} u_i [t] u_j [t] \Big) dt + V(t, x[t])_{t=\infty} \Big)$$

The validity of this expression can be verified by comparing the results of integration over t from 0 to ∞ of Eq. (1.16) and of the inequality

$$\frac{dV}{dt} \geqslant -F(t, x^*[t]) - \sum_{i,j=1}^{r} \beta_{ij} u_i^*[t] u_j^*[t]$$

where $u^*_j(t, x)$ are some functions which also solve the problem of stabilization of the motion $x = 0$ for Eqs. (1.5) for initial perturbations from domain (1.4) [1].

Example 1.1. Let us consider the system of differential equations [5]

$$\frac{dx_s}{dt} = \sum_{r=1}^{n} p_{sr} x_r$$

where the continuous bounded functions $p_{sr}(t)$ are such that

$$p_{sr} = -p_{rs} \ (r \neq s), \quad p_{ss} < -h = \text{const} < 0$$

for any $t \geqslant t_0$.

The asymptotic stability of the motion $x = 0$ can be verified by means of the Liapunov theorem by constructing the function

$$V = \frac{1}{2} \sum_{s=1}^{n} x_s^2, \qquad \frac{dV}{dt} = W = \sum_{s=1}^{n} p_{ss} x_s^2$$

The function V is optimal for the controlled system

$$\frac{dx_s}{dt} = \sum_{r=1}^{n} p_{sr} x_r + m_s u, \qquad J = \int_{t_0}^{\infty} \left(-\sum_{s=1}^{n} p_{ss} x_s^2 + \frac{1}{4\beta} \left(\sum_{s=1}^{n} m_s x_s \right)^2 + \beta u^2 \right) dt$$

where m_s are continuous functions of time, $\beta = \text{const} > 0$, and the controlling optimal force is

$$u^\circ = -{}^1/_2 \beta^{-1} (m_1 x_1 + \ldots + m_n x_n)$$

We note that the latter system is also asymptotically stable for $u = u^0$ and $m_1 = m(t)$, $m_s = 0$ $(s = 2, \ldots, n)$ when

$$p_{11} - {}^1/_2 m^2/\beta < -h, \quad p_{ss} < -h \ (s = 2, \ldots, n)$$

for any $t \geqslant t_0$.

Example 1.2. Let us consider the stabilization in a central force field of the circular motion of a material point controlled by a reactive force. The solution in linear approximation is obtained in [1]. Retaining the notation of [1], we can write the equations of perturbed motion of the point in the form

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = -\frac{\mu}{(r_0 + x_1)^2} + \frac{(\sqrt{\mu r_0} + x_3)^2}{(r_0 + x_1)^3} + bu$$

$$\frac{dx_3}{dt} = \frac{r_0 + x_1}{r_0} u, \qquad b = \frac{c_r}{c_\varphi r_0}$$

For $u = 0$ these equations have the first integrals

$$V_1 = x_2^2 + \frac{(\sqrt{\mu r_0} + x_3)^2}{(r_0 + x_1)^2} - \frac{2\mu}{r_0 + x_1} + \frac{\mu}{r_0} = \text{const}, \qquad V_2 = x_3 = \text{const}$$

from which the positive-definite Liapunov function

$$V = V_1 - 2\frac{\sqrt{\mu r_0}}{r_0^2} V_2 + \lambda V_2^2, \qquad W = 0 \qquad \left(\lambda = \text{const} > \frac{3}{r_0^2} \right)$$

can be obtained by the Chetaev method.

Setting $\omega(x, u) = F(x) + \beta u^2$ in (1.3), we obtain the optimal controlling force

$$u^\circ = -\frac{1}{2\beta}\left(\frac{\partial V}{\partial x_2} b + \frac{\partial V}{\partial x_3}\frac{r_0 + x_1}{r_0}\right) = -\frac{1}{\beta}\left[b x_2 + \left(\lambda + \frac{1}{r_0^3}\right) x_3 - \frac{2\sqrt{\mu r_0}}{r_0^3} x_1 + \ldots\right]$$

where the ellipsis represents second- and higher-order small terms and also the function

$$F(x) = \frac{1}{4\beta}\left(\frac{\partial V}{\partial x_2} b + \frac{\partial V}{\partial x_3}\frac{r_0 + x_1}{r_0}\right)^2$$

It is obvious that manifold (1.17) does not contain entire motions of the controlled motions under consideration for $\lambda > {}^3/r_0^3$ and $c_\varphi \neq 0$.

2. Let us consider the following equations of motion of a holonomic mechanical system in Lagrangian coordinates:

$$\frac{d}{dt}\frac{\partial L}{\partial q_i^\cdot} - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, \ldots, n) \tag{2.1}$$

whose equilibrium position is the point $q_i = 0,\ q_i^\cdot = 0$. The Lagrange function is not explicitly dependent on time and generally has the structure $L(q, q^\cdot) = L_2 + L_1 + L_0$.

In this case Eqs. (2.1) have the following (generalized) energy integral

$$H = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i^\cdot} q_i^\cdot - L = L_2 - L_0 = \text{const} \tag{2.2}$$

Let us suppose that the function $H(q, q^\cdot)$ is positive-definite in the neighborhood of the equilibrium position

$$|q_i| < h, \quad |q_i^\cdot| < h, \quad h = \text{const} > 0 \qquad (i = 1, \ldots, n) \tag{2.3}$$

The equilibrium position $q_i = q_i^\cdot = 0$ of the system is then stable. This position can be made asymptotically stable by subjecting the system to the additional forces

$$Q_i = \sum_{j=1}^{r} m_{ij}(q_s) u_j(q_s, q_s^\cdot), \qquad u_j(0,0) = 0 \tag{2.4}$$

so that the equations of motion become

$$\frac{d}{dt}\frac{\partial L}{\partial q_i^\cdot} - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{r} m_{ij} u_j \qquad (i = 1, \ldots, n) \tag{2.5}$$

By virtue of system (2.5), the time derivative of the function H is given by

$$\frac{dH}{dt} = \sum_{i=1}^{n} Q_i q_i^\cdot = \sum_{i=1}^{n}\sum_{j=1}^{r} m_{ij} u_j q_i^\cdot \tag{2.6}$$

If additional forces (2.4) are such that

$$Q_1 q_1^\cdot + \ldots + Q_n q_n^\cdot \leqslant 0 \tag{2.7}$$

and if the manifold

$$Q_1 q_1^\cdot + \ldots + Q_n q_n^\cdot = 0 \tag{2.8}$$

does not contain entire motions of system (2.5), then the equilibrium position becomes asymptotically stable [4].

Let us determine which controlling forces $u_j = u_j^0$ ensure the asymptotic stabilization of the equilibrium position with minimization of the functional

$$J=\int_{t_0}^{\infty}\left(F(q,q^{\cdot})+\sum_{i,j=1}^{r}\beta_{ij}u_iu_j\right)dt \tag{2.9}$$

where $F(q, q^{\cdot})$ is a nonnegative function to be determined, and where the quadratic form is some given positive-definite function of the controlling forces.

Let us write equations of the form (1.10) with the aid of the results of Sect. 1; they turn out to be

$$\sum_{i=1}^{n} m_{ij}q_i^{\cdot}+2\sum_{i=1}^{r}\beta_{ij}u_j^{\circ}=0 \qquad (j=1,\ldots,r)$$

and yield

$$u_j^{\circ}=-\frac{1}{2}\sum_{k=1}^{r}\frac{\Delta_{kj}}{\Delta}\sum_{i=1}^{n}m_{ik}q_i^{\cdot} \qquad (j=1,\ldots,r) \tag{2.10}$$

i.e. the controlling forces are linear functions of the generalized velocities.

Replacing the u_j in the expression of the form (1.9) by the above values and equating the result to zero, we obtain an equation which enables us to find an expression for the function

$$F(q,q^{\cdot})=\sum_{i,j=1}^{r}\beta_{ij}u_i^{\circ}u_j^{\circ} \tag{2.11}$$

which is a quadratic form of the generalized velocities.

This form, which is positive-definite in u_i°, is generally positive-constant in the generalized velocities $q_i^{\cdot}$. However, if the coefficients m_{ik} of additional forces (2.4) are such that the equations

$$\sum_{k=1}^{r}\Delta_{kj}\sum_{i=1}^{r}m_{ik}q_i^{\cdot}=0 \qquad (j=1,\ldots,r)$$

have only the trivial solution $q_i^{\cdot}=0$ in domain (2.3), then form (2.11) is a positive-definite function of $q_i^{\cdot}$.

By virtue of Eqs. (2.5) the time derivative of the system energy H for $u_j=u_j^{\circ}$ is given by

$$\frac{dH}{dt}=-2\sum_{i,j=1}^{r}\beta_{ij}u_i^{\circ}u_j^{\circ}=-2F(q,q^{\cdot})$$

where u_j° are defined by Eqs. (2.10); moreover, we clearly have

$$Q_i^{\circ}=\sum_{j=1}^{r}m_{ij}u_j^{\circ}=-\frac{\partial F}{\partial q_i^{\cdot}}$$

so that the quadratic form $F(q, q^{\cdot})$ of the generalized velocities can be considered as a Rayleigh dissipative function; the additional forces Q_i° can be assumed to belong to the class of dissipative forces. The dissipation in this case is either complete or partial depending on whether function (2.11) is a positive-definite function or a positive-constant function of $q_i^{\cdot}$. In the latter case manifold (2.8) is of the form

$$\sum_{i,j=1}^{r}\beta_{ij}u_i^{\circ}u_j^{\circ}=0 \tag{2.12}$$

We note that in the case of a scalar control where $u_j=u$,

$$\sum_{j=1}^{r}m_{ij}=m_i, \qquad \sum_{i,j=1}^{r}\beta_{ij}=\beta>0$$

instead of (2.10), (2.11) we have [6, 7]

$$u^\circ = -\frac{1}{2\beta}\sum_{i=1}^{n} m_i q_i^{\cdot}, \quad F = \beta u^{\circ 2}$$

Specifically, if the system is controlled with respect to the first coordinate only, then $m_1 = m,\ m_s = 0\ (s = 2, \ldots, n)$, so that

$$u^\circ = -\frac{m}{2\beta} q_1^{\cdot}, \qquad F = \frac{m^2}{4\beta} q_1^{\cdot 2}$$

and functional (2.9) becomes

$$J = \int_{t_0}^{\infty}\left(\frac{m^2}{4\beta} q_1^{\cdot 2} + \beta u^2\right) dt$$

Let us formulate our result.

If the energy H (2.2) for system (2.1) is a positive-definite function of the generalized coordinates and velocities, then it is the optimal Liapunov function for system (2.5) optimized by controlling forces (2.10) with respect to functional (2.9), (2.11) under the condition that manifold (2.12) does not contain entire motions of system (2.5) other than $q_i = q_i^{\cdot} = 0$.

This result complements the results of [6, 7]. The conditions of stabilization by dissipative forces were investigated in [5, 8].

Example 2.1. Let us consider the optimal stabilization of the positions of relative equilibrium of a triaxial gyrostat satellite [9] in an orbital coordinate system uniformly rotating about the y-axis at the Keplerian angular velocity ω_0.

Let us consider some stable position of relative equilibrium for which the generalized coordinates q_i $(i = 1, 2, 3)$ defining the position of the satellite body in the orbital coordinate system have the values $q_i = q_{i0}$ and in whose neighborhood the energy H is a positive-definite function of q_i, $q_i^{\cdot}$. We take optimizable functional (2.9) in the form

$$J = \int_{t_0}^{\infty}\left(F(q, q^{\cdot}) + \sum_{i=1}^{3} \beta_i u_i^2\right) dt$$

where $\beta_i > 0$ $(i = 1, 2, 3)$ and assume that the coefficients of the controlling forces in expressions (2.4) satisfy the conditions

$$m_{ii} = m_i, \qquad m_{ij} = 0 \qquad (i \neq j)$$

In accordance with formulas (2.10) and (2.11) we obtain

$$u_i^\circ = -\frac{m_i}{2\beta_i} q_i^{\cdot} \quad (i = 1, 2, 3), \qquad F(q, q_i^{\cdot}) = \sum_{i=1}^{3} \frac{m_i^2}{4\beta_i} q_i^{\cdot 2}$$

Such controlling forces constitute dissipative forces with complete dissipation; they ensure optimal stabilization of the stable relative equilibrium of a satellite in whose neighborhood the function H is positive-definite, i.e. for any point of the domain of fulfillment of the sufficient conditions of stability. At the same time such forces disrupt the stability achieved through the action of gyroscopic forces [5, 6].

Certain cases allow optimal stabilization of the equilibrium positions of a satellite and forces with partial dissipation. To be specific, let us consider the position of relative equilibrium in which the principal inertial axes x_1, x_2, x_3 of the satellite coincide, respectively, with the axes $z, -x, -y$ of the orbital coordinate system where the constant gyrostatic moment $\mathbf{k}$ of the gyro wheels is directed along the axis x_3. The position of the satellite in the orbital system is defined by the Euler angles θ, ψ, φ which have the values $\theta_0 = {}^1/_2\pi$; $\psi_0 = 0$, $\varphi_0 = {}^1/_2\pi$ for the equilibrium position in question. The sufficient conditions of relative equilibrium of the satellite with respect to the Euler angles and their time derivatives are readily obtainable with the aid of the Lagrange theorem and turn out to be

$$C - k/\omega_0 > B > A, \qquad C - {}^1/_4 k/\omega_0 > A$$

where A, B, C are the principal central moments of inertia of the gyrostat.

Let us consider the case of controlling forces with partial dissipation, where either $m_1 = 0$ or $m_2 = 0$ and, respectively, $\beta_1 = 0$ or $\beta_2 = 0$, but $m_3 \neq 0$, $\beta_3 \neq 0$.

From the equations in variations for perturbed motion, namely

$$B\xi_1^{\cdot\cdot} + \left(A + B - C + \frac{k}{\omega_0}\right)\omega_0\xi_2^{\cdot} + \left(4(C - A) - \frac{k}{\omega_0}\right)\omega_0^2\xi_1 = -\frac{m_1^2}{2\beta_1}\xi_1^{\cdot}$$

$$A\xi_2^{\cdot\cdot} - \left(A + B - C + \frac{k}{\omega_0}\right)\omega_0\xi_1^{\cdot} + \left(C - B - \frac{k}{\omega_0}\right)\omega_0^2\xi_2 = -\frac{m_2^2}{2\beta_2}\xi_2^{\cdot}$$

$$C\xi_3^{\cdot\cdot} + 3\omega_0^2(B - A)\xi_3 = -\frac{m_3^2}{2\beta_3}\xi_3^{\cdot}$$

(where the ξ_i denote the variations of the angles θ, ψ, φ) we see that optimal stabilization of stable relative equilibrium is ensured for both $m_1 = 0$ $(m_2 \neq 0, m_3 \neq 0)$ and $m_2 = 0$ $(m_1 \neq 0, m_3 \neq 0)$ if

$$A + B - C + k/\omega_0 \neq 0$$

3. Let us consider the problem of optimal stabilization of the motion of a system with respect to some $(x_1, \ldots, x_k)$ rather than all $(x_1, \ldots, x_n)$ of the variables characterizing the system. This problem, like that of stability of motion with respect to some of the variables [3], is of interest in many practical cases.

For brevity we write the equations of perturbed motion of the controlled system in vector form ,

$$dx / dt = X(t, x, u) \tag{3.1}$$

where $x = (y_1, \ldots, y_k, z_1, \ldots, z_m)$ is the real n-vector of state of the system, where $n = k + m$, $k > 0$, $m \geqslant 0$, $u = (u_1, \ldots, u_r)$ is the real control r-vector $r > 0$. We assume that the real n-vector function $X(t, x, u)$ is defined and continuous in the domain

$$t \geqslant t_0, \quad |y_i| \leqslant H, \quad z_j \text{ are arbitrary} \tag{3.2}$$

for all possible values of the control vector u sought in the form of the r-vector function $u(t, y, z)$ which must be defined and continuous in domain (3.2). We assume that the vector functions X and u satisfy the conditions ensuring the existence and uniqueness of the solutions of Eq. (3.1) for all initial conditions from domain (3.2); moreover, each solution of this equation is z-continuable (in other words, every z-component of the solution of Eq. (3.1) continues to be defined as long as $|y_s| \leqslant H$). We also assume fulfillment of the identities

$$X(t, 0, 0) = 0, \quad u(t, 0, 0) = 0$$

i.e. that Eq. (3.1) has the solution $x = 0$. The control quality criterion can be expressed as the minimum condition for the integral

$$J = \int_{t_0}^{\infty} \omega(t, x[t], u[t])\, dt \tag{3.3}$$

where $\omega\ (t,\ x,\ u)$ is some nonnegative scalar function defined in domain (3.2).

Problems on stabilization and optimal stabilization with respect to part of the variables which generalize the corresponding problems [1] for all the variables can be formulated for system (3.1).

Let us state the problem of optimal stabilization with respect to some of the variables.

We are to find the vector of controlling forces $u^\circ\ (t,\ x)$ which ensures the asymptotic stability of the unperturbed motion $x = 0$ with respect to the y-component of the vector x (y-asymptotic stability) by virtue of Eq. (3.1) (for $u = u^\circ (t,\ x)$). Whatever the other vector of controlling forces $u^*\ (t,\ x)$ ensuring the y-asymptotic stability of the motion $x = 0$, we necessarily have the inequality

$$\int_{t_0}^{\infty} \omega\,(t, x^\circ [t]; u^\circ [t])\, dt \leqslant \int_{t_0}^{\infty} \omega\,(t, x^* [t], u^* [t])\, dt \tag{3.4}$$

for all initial conditions t_0, $x\ (t_0)$ from the domain

$$t \geqslant 0, \qquad |x_s\,(t_0)| \leqslant \lambda \tag{3.5}$$

where the positive constant λ is either prescribed in the conditions of the problem or has the same meaning as in the proof of the theorem on stability with respect to some of the variables [3].

The meaning of the problem of y-optimal stabilization makes it expedient to define the function ω in (3.3) and the control vector independently of the z-component of the vector x ; however, since the former may, in fact, depend on the z-component, we have stated the problem in its general form.

Let us recall some definitions. The fixed-sign function $V\ (t,\ x)$ is called "y-positive-definite" if there exists a time-independent positive-definite function $W\ (y)$ such that the difference $V - W \geqslant 0$ in domain (1.2).

The function $V\ (t,\ x)$ bounded in domain (1.2) admits an infinitely small upper bound in y if for any arbitrarily small positive number l there exists a number $\lambda > 0$ such that $|V| \leqslant l$ for arbitrary $t \geqslant t_0$, $|y_i| \leqslant \lambda$, z_j . If the former inequality is fulfilled for $t \geqslant t_0$, $|x_i| \leqslant \lambda$, we say that the function V can have an infinitely small upper bound (in all its variables).

As in the case of stability with respect to all the variables, there exist several formulations of the theorem on asymptotic stability with respect to some of the variables. This in turn means that there are several possible variants of the theorem on optimal stabilization with respect to some of the variables. We confine our attention to two of these variants (*) [3, 10].

T h e o r e m 3.1 (on y-optimal stabilization). If the differential equations of perturbed motion (3.1) are associated with a y-positive-definite function $V^\circ\ (t,\ x)$ which admits an infinitely small upper bound in y (in all the variables x) and with a vector function $u^\circ\ (t,\ x)$ satisfying the following conditions in domain (3.2):

---

*) We take this opportunity to refine the formulation of Theorem 2 of [3]: according to the proof given in [3] the function $V\ (t,\ x_1,\ldots,\ x_n)$ must admit of an infinitely small upper bound in $x_1,\ldots,\ x_m$. This proof is also valid if $V\ (t,\ x_1,\ldots,\ x_n)$ admits an infinitely small upper bound in $x_1,\ldots,\ x_k$ provided $V^\cdot$ is a fixed-sign function in the variables $x_1,\ldots,\ x_k\ (m \leqslant k \leqslant n)$.

1) the function

$$w(t, x) = \omega(t; x; u^\circ(t, x))$$

is positive-definite in y (in all the variables);

2) the equation

$$B[V^\circ;\ t;\ x;\ u^\circ(t,\ x)] = 0 \tag{3.6}$$

is valid;

3) the inequality

$$B[V^\circ;\ t;\ x;\ u] \geqslant 0 \tag{3.7}$$

holds whatever the numbers u_j, then the vector function $u^\circ(t,\ x)$ solves the problem of y-optimal stabilization. Moreover,

$$\int_{t_0}^{\infty} \omega(t; x^\circ[t]; u^\circ[t])\, dt = \min \int_{t_0}^{\infty} \omega(t; x[t]; u[t])\, dt = V^\circ(t_0, x(t_0)) \tag{3.8}$$

Proof. All of the conditions of the theorem [3] on stability in y, i.e. $|y_i| < A$ for $t \geqslant t_0$ if $|x_{s0}| \leqslant \lambda$, are fulfilled for $u = u^\circ(t, x)$. Here $0 < A < H$ is an arbitrary number and $\lambda > 0$ is a number defining the domain free of all points of the surface $V^\circ(t_0, x) = l$, where l is the exact lower bound of the function $W(y)$ under the condition $\|y\| = A$.

It is easy to see that

$$\lim V^\circ(t,\ x[t]) = 0,\ t \to \infty$$

for all initial values x_{s0} lying in the domain $|x_{s0}| \leqslant \lambda$.

Since $V(t,\ x[t])$ is a monotonic nonincreasing function, it tends to some limit e, $V^\circ \geqslant e$, as $t \to \infty$. Let us suppose that $e > 0$. It would then follow by the property of the function V°, which can have an infinitely small upper bound in y (in x), that there exists a number ε defining the domain $|y| \leqslant \varepsilon$ ($|x| \leqslant \varepsilon$) for whose points the values of V are smaller than e. The values of the variables y_s (x_i) would then lie somewhere in the domain

$$\varepsilon \leqslant |y| \leqslant A \qquad (\varepsilon \leqslant |x|)$$

Let us denote the lower bound of the function $-dV^\circ/dt$ in this domain by $l' > 0$. Then for any $t \geqslant t_0$ the values of the function V° in this domain satisfy the condition $dV^\circ/dt \leqslant -l'$. The equation

$$V^\circ - V_0^\circ = \int_{t_0}^{t} \frac{dV^\circ}{dt}\, dt$$

then implies that

$$V^\circ \leqslant V_0^\circ - l'(t - t_0)$$

which is impossible, since the left side of the inequality is a positive-definite function in y, and since the right side becomes negative for sufficiently large t. Hence, $e = 0$, i.e. $\lim y = 0$ as $t \to \infty$. We have therefore proved that the controlling forces $u^\circ(t, x)$ ensure the asymptotic stability in y of the motion $x = 0$. Now let us verify the validity of relation (3.8) by the method of [1]. Integrating the equation

$$dV^\circ/dt = -\omega(t,\ x,\ u^\circ)$$

(which follows from condition (3.6)) along the motion $x^\circ[t]$ from t_0 to ∞ and recalling that

$$\lim V^\circ(t,\ x^\circ[t]) = 0 \quad \text{as} \quad t \to \infty$$

we obtain

$$V^\circ(t_0,\ x(t_0)) = \int_{t_0}^{\infty} \omega(t;\ x^\circ[t];\ u^\circ[t])\, dt \tag{3.9}$$

On the other hand, let $u^*(t,x)$ be some control which also ensures the stabilization in y of the motion $x=0$ for initial perturbations from the domain $|x_{s0}| \leqslant \lambda$. Then by vertue of (3.7) we have the inequality

$$dV^\circ/dt \geqslant -\omega(t,\ x^*[t];\ u^*[t])\, dt$$

Integrating this inequality over time from t_0 to ∞ and again recalling the limit relation for V^0, we obtain

$$V^\circ(t_0,\ x(t_0)) \leqslant \int_{t_0}^{\infty} \omega(t;\ x^*[t];\ u^*[t])\, dt$$

Comparing this inequality with Eq. (3.9), we conclude that condition (3.8) is valid. The theorem has been proved.

Note 3.1. (1) The theorem remains valid for controlling forces $u = u(t)$ and also in the case where the controlling forces are restricted by some additional inequalities $|u| \leqslant a$. In the latter case we merely require that condition (3.7) be fulfilled for all values of u under the prescribed restrictions [1]; (2) the theorem is valid for the problem of y-optimal stabilization in the whole if we require fulfillment of conditions (3.6) and (3.7) for all y_i $(-\infty < y_i < \infty)$ and if $V^\circ(t,\ x) \to \infty$ for $y \to \infty$.

4. Let us reconsider the equations of motion of a holonomic mechanical system. Let us suppose that the coordinates q_α $(\alpha = k+1, \ldots, n)$ are cyclical, i.e. that

$$\partial L / \partial q_\alpha = 0 \qquad (\alpha = k+1, \ldots, n)$$

Equations (2.1) then have the first integrals

$$p_\alpha = \partial L / \partial q_\alpha^{\cdot} = c_\alpha \qquad (\alpha = k+1, \ldots, n) \tag{4.1}$$

Let Eqs. (2.1) have the particular solution

$$q_j = 0, \quad q_j^{\cdot} = 0 \quad (j = 1, \ldots, k), \qquad q_\alpha^{\cdot} = \omega_\alpha = \text{const} \tag{4.2}$$

describing steady motion for certain fixed values of the constants $c_\alpha = c_\alpha^\circ$. The latter can be stable with respect to the variables q_j, $q_j^{\cdot}$, $q_\alpha^{\cdot}$ only; it is unstable with respect to the variables q_α. Ignoring the cyclical coordinates by the Routh method, we introduce the Routh function

$$R(q_j,\ q_j^{\cdot},\ c_\alpha) = L - \sum_{\alpha=k+1}^{n} q_\alpha^{\cdot} c_\alpha$$

and write the equations of motion in the form

$$\frac{d}{dt}\frac{\partial R}{\partial q_j^{\cdot}} - \frac{\partial R}{\partial q_j} = 0 \quad (j = 1, \ldots, k) \tag{4.3}$$

which constitute the Lagrange equations for the reduced system.

Equations (4.3) constitute the equations of perturbed motion for steady motion (4.2); once they have been integrated the cyclical coordinates q_α can be found by quadratures. Let us suppose that the function R is not explicitly dependent on time; Eqs. (4.3) then have the energy integral

$$H = \sum_{j=1}^{k} \frac{\partial R}{\partial q_j^{\cdot}} q_j^{\cdot} - R = R_2 - R_0 = \text{const} \tag{4.4}$$

If the function H is a positive-definite function of the variables q_j, $q_j^{\cdot}$, then motion (4.2) is stable with respect to these variables and also with respect to p_α, and therefore with respect to $q_\alpha^{\cdot}$. Now let us subject the system to the controlling forces

$$Q_j = m_{j1}u_1 + \ldots + m_{jr}u_r \tag{4.5}$$

as a result of which the equations of perturbed motion become

$$\frac{d}{dt}\frac{\partial L}{\partial q_j^{\cdot}} - \frac{\partial L}{\partial q_j} = Q_j, \qquad \frac{\partial L}{\partial q_\alpha^{\cdot}} = c_\alpha \tag{4.6}$$

or, which is the same thing,

$$\frac{d}{dt}\frac{\partial R}{\partial q_j^{\cdot}} - \frac{\partial R}{\partial q_j} = \sum_{s=1}^{r} m_{js}u_s \qquad (j = 1, \ldots, k) \tag{4.7}$$

similar in form to Eqs. (2. 5). Introducing an integral of the form (2. 9) and repeating the argument of Sect. 2, we conclude that the result of Sect. 2 is valid for reduced system (4. 7), (2. 9). But optimal stabilization of the reduced system in the variables q_j, $q_j^{\cdot}$ is equivalent to optimal stabilization of initial system (4. 6), (2. 9) in some of the variables.

This implies the validity of the follwing theorem.

Theorem 4.1. If the function H (4. 4) is a positive-definite function of the positional coordinates and velocities, then it constitutes the optimal Liapunov function for system (4. 6) optimized in the variables q_j, $q_j^{\cdot}$ with respect to functional (2. 9), (2. 11) by controlling forces (2. 10) provided manifold (2. 12) does not contain entire motions of system (4. 7).

Note 4.1. Since $H = R_2(q_j, q_j^{\cdot}) + W(q_j, c_\alpha)$, where $R_2(q_j, q_j^{\cdot})$ is a fixed-sign function in $q_j^{\cdot}$, then H is of fixed sign provided the function $W(q_j, c_\alpha) = -R_0$ is positive-definite in q_j, i. e. if it has a minimum for $q_j = 0$.

Example 4.1. Let us consider in restricted formulation the problem of optimal stabilization of the equilibrium position in the orbital coordinate system of the axis of symmetry of a symmetric gyrostat satellite rotating at some angular velocity about its axis of symmetry [9]. We define the position of the gyrostat body in the orbital system by the Euler angles θ, ψ, φ; we also assume that the gyrostatic moment is colinear with the axis of symmetry of the satellite. The proper rotation angle φ is a cyclical coordinate; ignoring the latter, we obtain the Routh function

$$\begin{aligned} R(\theta, \psi, \theta^{\cdot}, \psi^{\cdot}, c) = {}& {}^1/_2 A\,[\theta^{\cdot 2} + \psi^{\cdot 2}\sin^2\theta + 2\omega_0(\psi^{\cdot}\sin\theta\cos\theta\cos\psi + \theta^{\cdot}\sin\psi)] + \\ & + c\psi^{\cdot}\cos\theta - {}^3/_2\,\omega_0^2(C - A)\cos^2\theta - {}^1/_2 A\omega_0^2\sin^2\theta\cos^2\psi - \\ & - c\omega_0\sin\theta\cos\psi - {}^1/_2(c-k)^2 / C \end{aligned} \tag{4.8}$$

where c is the constant associated with the cyclical integral,

$$C(\varphi^{\cdot} + \psi^{\cdot}\cos\theta - \omega_0\sin\theta\cos\psi) + k = c$$

The equations of motions have the solutions

$$\theta = \theta_0, \ \psi = \psi_0, \ \theta^{\cdot} = \psi^{\cdot} = 0 \tag{4.9}$$

which break down into the following three families:

$$\theta_0 = {}^1/_2\pi, \qquad \psi_0 = \pi \tag{4.10}$$

$$\theta_0 = {}^1/_2\pi, \qquad \cos\psi_0 = -c/A\,\omega_0 \tag{4.11}$$

$$\sin\theta_0 = c/(4A - 3C)\,\omega_0^{-1}, \ \psi_0 = \pi \tag{4.12}$$

Solutions (4. 9) describe the steady motions of a gyrostat satellite in which the axis of symmetry is either colinear with the orbital plane ((4.10)) or orthogonal to the radius vector of the center of mass ((4.11)) or orthogonal to the tangent to the orbit ((4.12)).

Let us consider some stable steady motion for which (4. 9) have specific values and in whose neighborhood energy (4. 4) is a positive-definite function of the variations θ, ψ, $\theta^{\cdot}$, $\psi^{\cdot}$. We take optimizable functional (2. 9) in the form

$$J=\int_{t_0}^{\infty}(F(\theta,\ \psi,\ \theta^{\cdot},\ \psi^{\cdot})+\beta_1 u_1^2+\beta_2 u_2^2)\,dt$$

where $\beta_i > 0$ $(i = 1, 2)$, assuming that the coefficients of the controlling forces in expressions (4.5) satisfy the conditions

$$m_{ii} = m_i, \qquad m_{ij} = 0 \quad (i \neq j)$$

In accordance with formulas (2.10) and (2.11) we obtain

$$u_1^{\circ}=-\frac{m_1}{2\beta_1}\theta^{\cdot}, \quad u_2^{\circ}=-\frac{m_2}{2\beta_2}\psi^{\cdot}, \quad F=\frac{1}{4}\left(\frac{m_1^2}{\beta_1}\theta^{\cdot 2}+\frac{m_2^2}{\beta_2}\psi^{\cdot 2}\right) \tag{4.13}$$

Controlling forces of this form for the reduced system constitute dissipative forces with complete dissipation; they ensure optimal stabilization with respect to the variables θ, ψ, $\theta^{\cdot}$, $\psi^{\cdot}$ of the stable steady motion in whose neighborhood the function H is positive-definite.

Now let us consider forces with partial dissipation, when either $m_1 \neq 0$, $m_2 = \beta_2 = 0$ or $m_2 \neq 0$, $m_1 = \beta_1 = 0$ in (4.13). The equations of motion in the neighborhood of steady motion (4.9) can be expressed in variations in the form

$$\xi_1^{\cdot\cdot}-\Omega\xi_2^{\cdot}+a_1\xi_1=-\frac{m_1}{2\beta_1}\xi_1^{\cdot}, \quad \xi_2^{\cdot\cdot}+\Omega\xi_1^{\cdot}+a_2\xi_2=-\frac{m_2}{2\beta_2}\xi_2^{\cdot} \tag{4.14}$$

where ξ_i are the variations of the angles θ and ψ; Ω and a_i are constant coefficients given by the following expressions:

for solution (4.10),

$$\Omega = 2 - a, \qquad a_1 = a + b - 1, \qquad a_2 = a - 1 \tag{4.15}$$

for solution (4.11)

$$\Omega = a, \qquad a_1 = b, \qquad a_2 = 1 - a^2 \tag{4.16}$$

where

$$a = c/A\omega_0, \qquad b = 3\,(C/A - 1) \tag{4.17}$$

The equations in variations for solution (4.12) are of the form

$$\xi_1^{\cdot\cdot}-\frac{(1+b)}{(1-b)^2}a^2\xi_2^{\cdot}+\frac{(1-b)^2-a^2}{1-b}\xi_1=-\frac{m_1}{2\beta_1}\xi_1^{\cdot}$$

$$\xi_2^{\cdot\cdot}+(1+b)\,\xi_1^{\cdot}-b\xi_2=-\frac{m_2}{2\beta_2}\xi_2^{\cdot} \tag{4.18}$$

The equations in variations imply that optimal stabilization in the variables θ, ψ, $\theta^{\cdot}$, $\psi^{\cdot}$ of steady motion (4.9) stable for $a_1 > 0$, $a_2 > 0$ is ensured by controlling forces of the form (4.13) for both $m_1 \neq 0$, $m_2 = 0$ and $m_1 = 0$, $m_2 \neq 0$ under the condition $\Omega \neq 0$ (for solutions (4.10) and (4.11)) or $b \neq -1$ (for solution (4.12)). The shaded

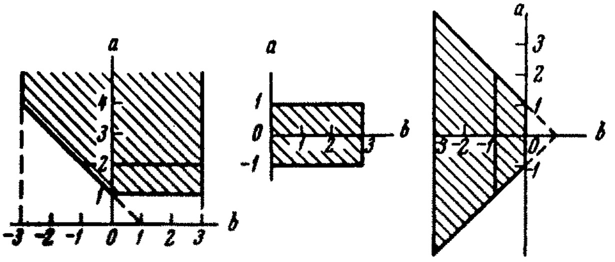

Fig. 1

areas in Fig. 1 represent the domains of optimal stabilization by forces with partial dissipation corresponding to Eqs. (4.10)–(4.12) and coincident with the domains of fulfillment of the sufficient conditions of stability with the exception of the straight line $a = 2$ for solution (4.10), the straight line $a = 0$ for solution (4.11), and the straight line $b = -1$ for solution (4.12) on which the conditions of optimal stabilization are not fulfilled in the first approximation. However, consideration of the nonlinear terms in the equations of perturbed motion indicate that the conditions of optimal stabilization are also fulfilled on these straight lines.

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ON THE PRECISION OF OPTIMAL CONTROL OF THE FINAL STATE

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The Bellman partial differential equation involved in the synthesis of a stochastically optimal control of the final state of a linear system is considered. Approximate formulas and estimates of the solution are derived on the basis of the solution of the Bellman equation for the determinate variant of the problem. A numerical method of solution is proposed. The problem in the one-dimensional case is reduced to an integral equation of the first kind; a finite formula for the solution is derived under certain additional assumptions.